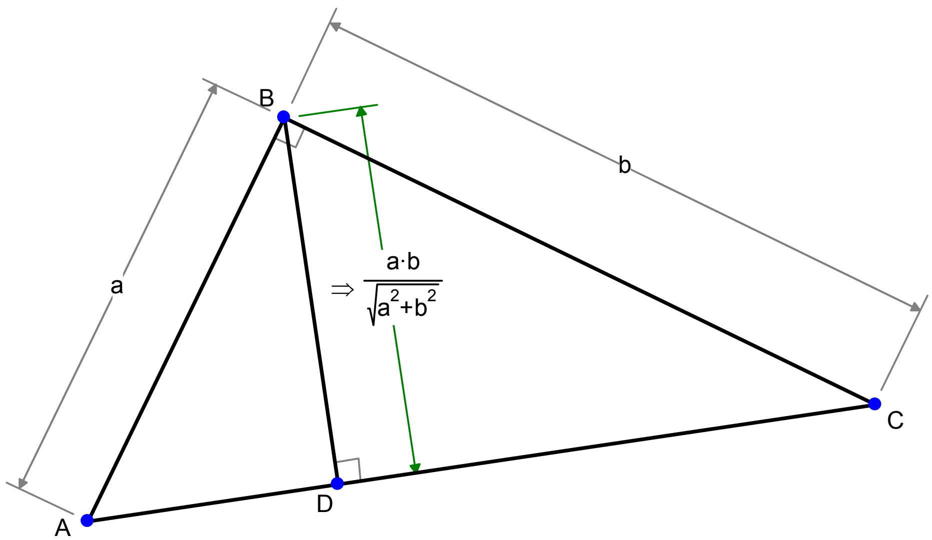
Conics with Geometry Expressions

Introduction

Geometry Expressions automatically generates algebraic expressions from geometric figures. For example in the diagram below, the user has specified that the triangle is right and has short sides length a and b. The system has calculated an expression for the length of the altitude:

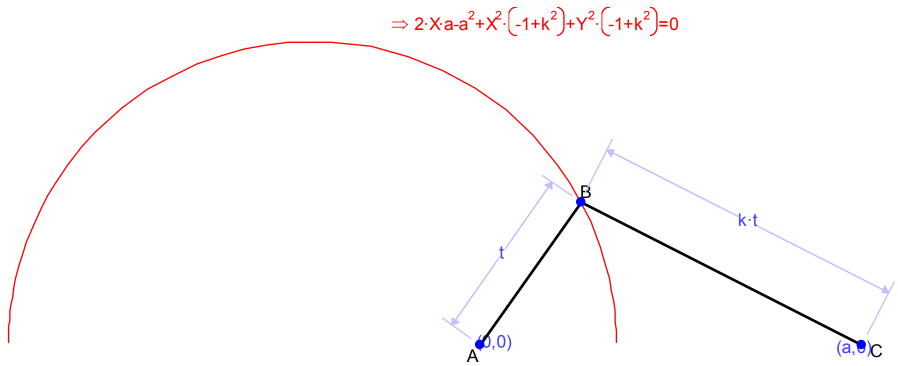


In this article, we create a set of examples investigating conics with this tool.

Although Version 1 of Geometry Expressions does not have conics (other than circles) as built in types, they can be studied using loci.

Example 1: Circle of Apollonius

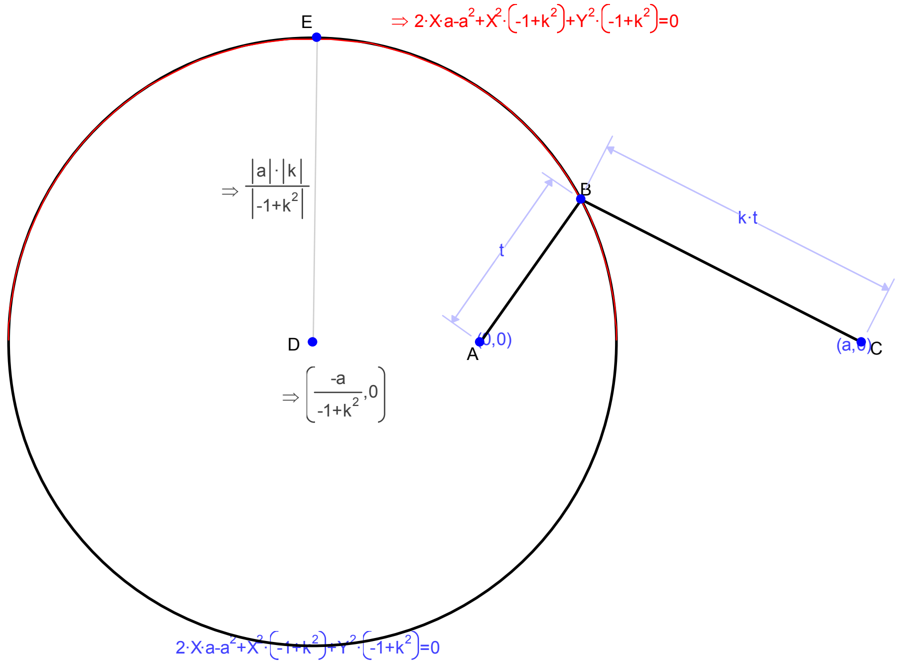
The Circle of Apollonius is the locus of points the ratio of whose distance from a pair of fixed points is constant:



How do we know this is a circle?

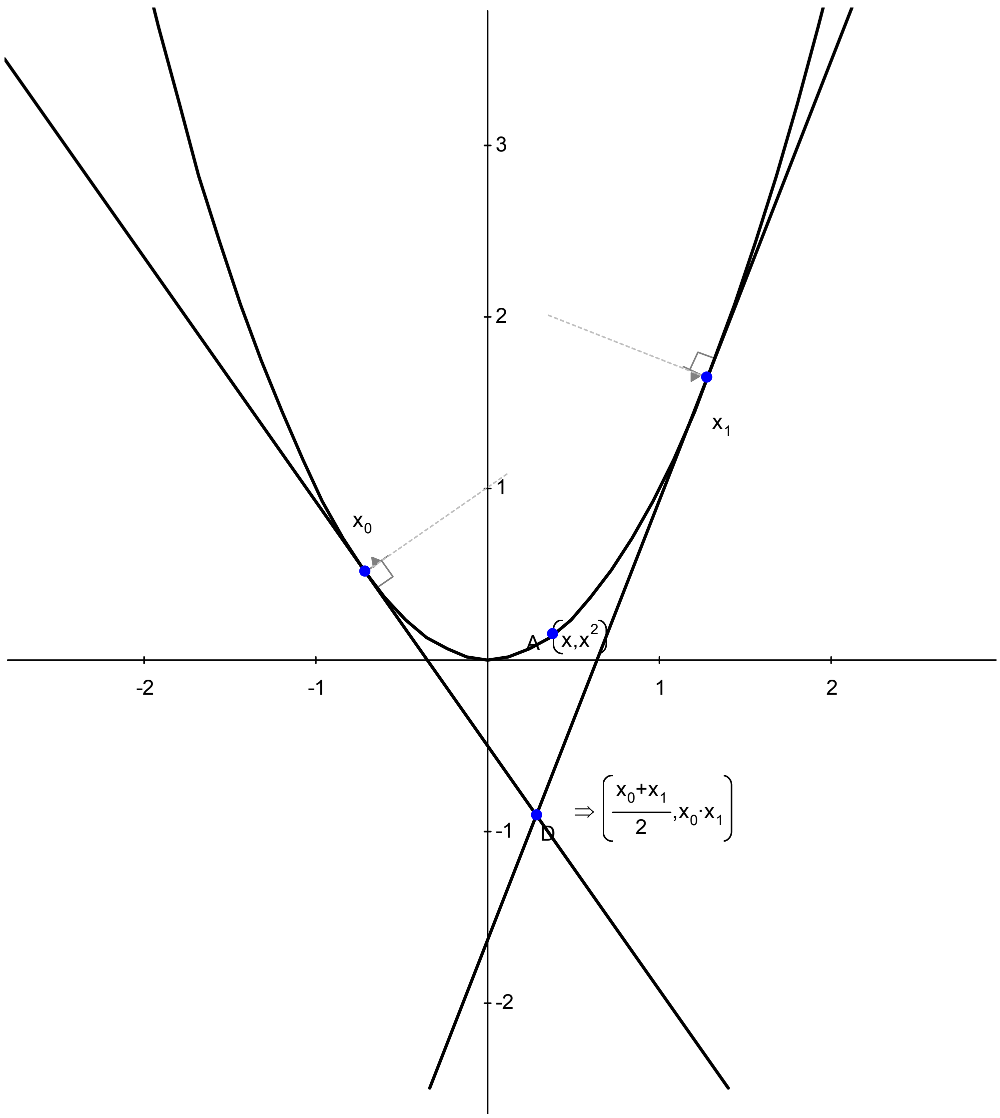
What is the center and radius?

You can always get Geometry Expressions to tell you: draw a circle and set its equation to be the same as the locus equation (copy and paste works fine)



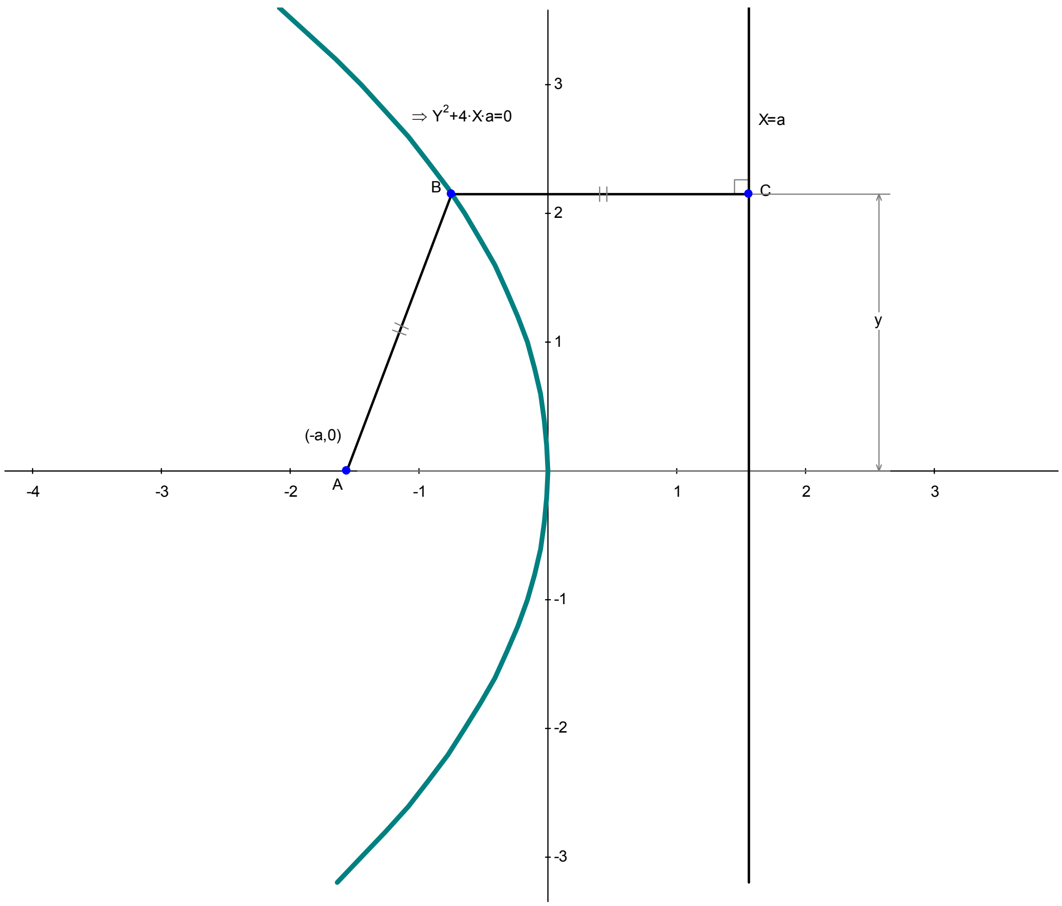
Example 2: Intersection of two tangents to the curve y=x^2

We create the point (x,x^2) and draw its locus as x goes from -3 to 3. Now we create two tangents to the curve, and examine their intersection.



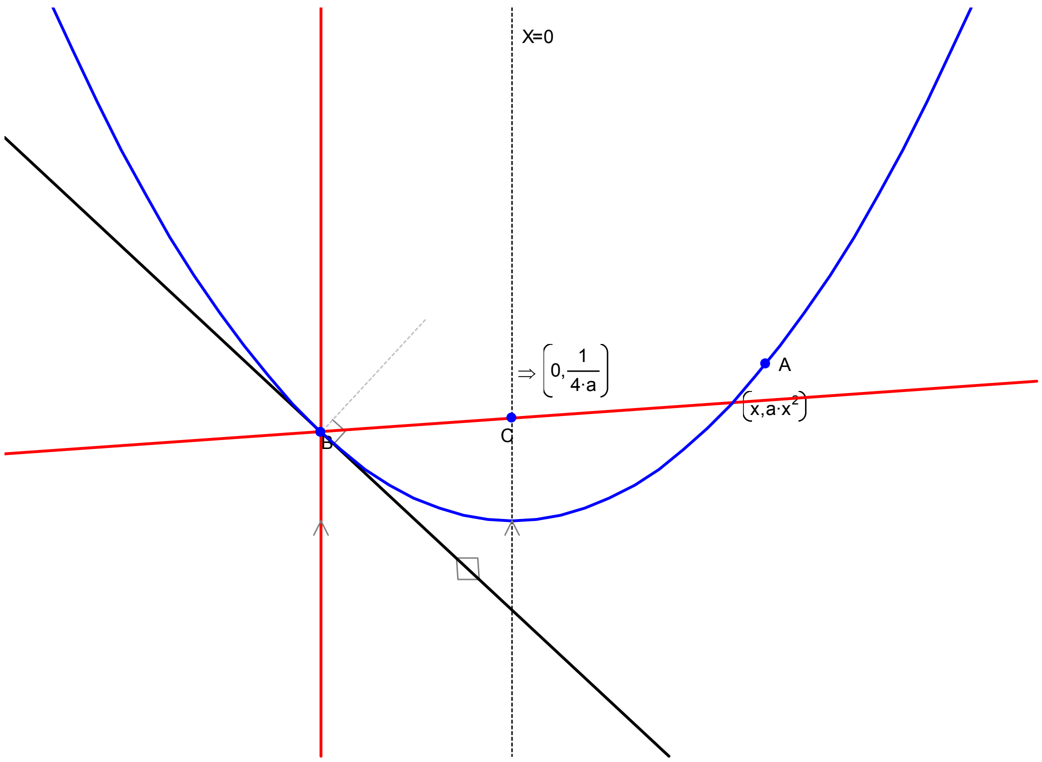
Example 3: Parabola as locus of points equidistant between a point and a line

Here is the equation of the parabola which is the locus of points equidistant from the point (-a,0) and the line X=a:



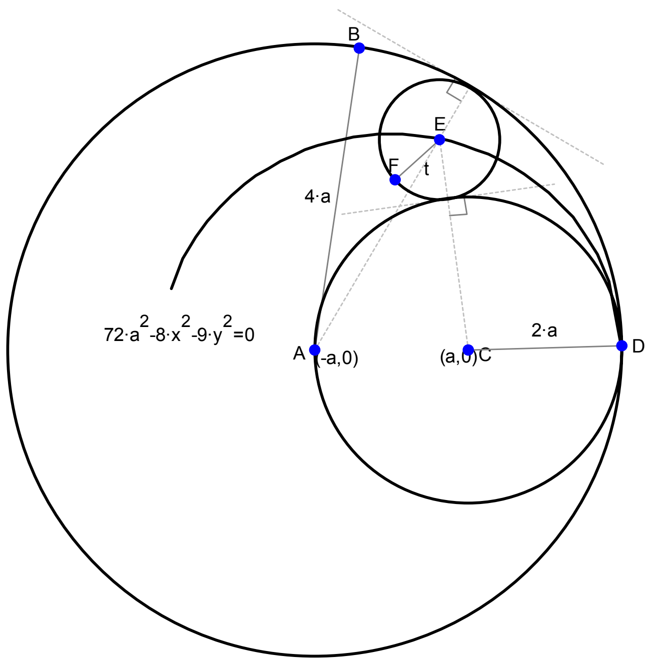
Example 4: Parabolic Mirror

A parabolic mirror focuses parallel rays into a single point. Where is that point? We create the parabola $y = a \cdot x^2$ and reflect a ray parallel to the y axis in the the tangent to the curve. We examine the y intercept of the image:



Example 5: Squeezing a circle between two circles

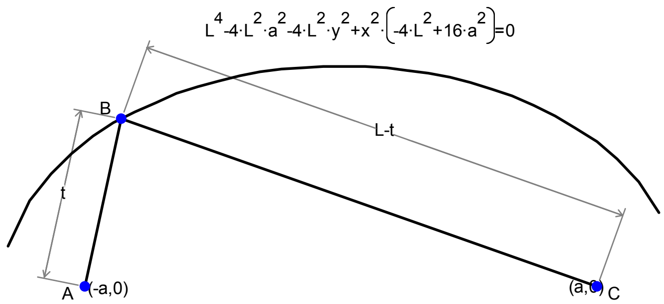
Take a circle radius 2a centered at (a,0) and a circle radius 4a centered at (-a,0). Now look at the locus of the center of the circle tangent to both.



It's an ellipse. From the drawing we can see that the semi major axis in the x direction is 3a. What is the semi major axis in the y direction?

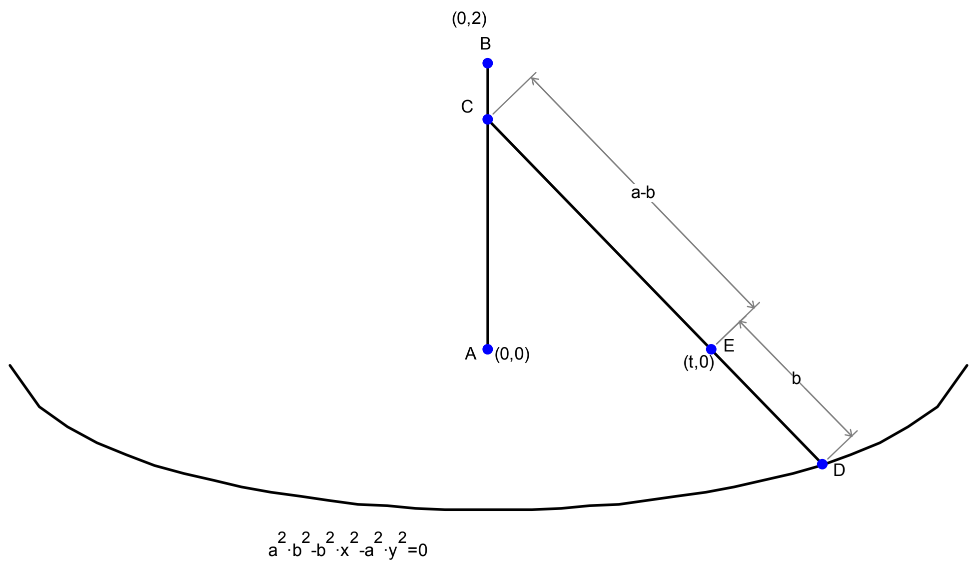
Example 6: Ellipse as a locus

Here is the usual string based construction of an ellipse foci (-a,0) (a,0):



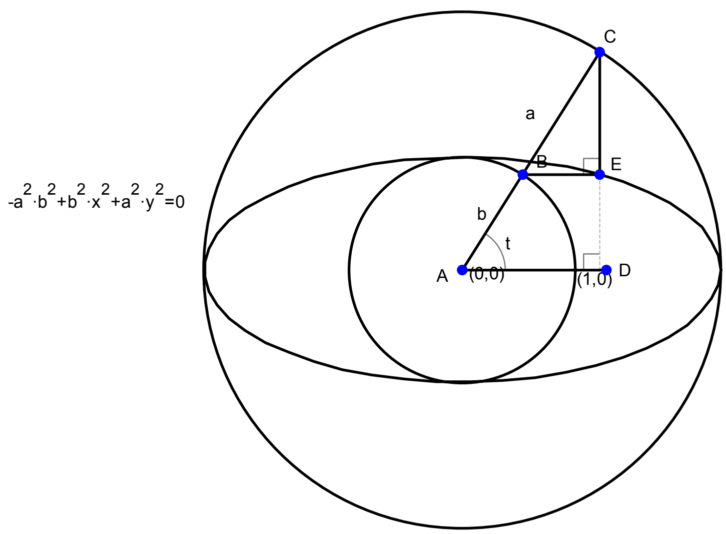
Example 7: Archimedes Trammel

A mechanism which generates an ellipse is Archimedes Trammel. The points C and E are constrained to run along the axes, while the distance between them is set to a-b. We trace the locus of the point D distance b from E along the same line. This gives an ellipse with semi major axes a and b:



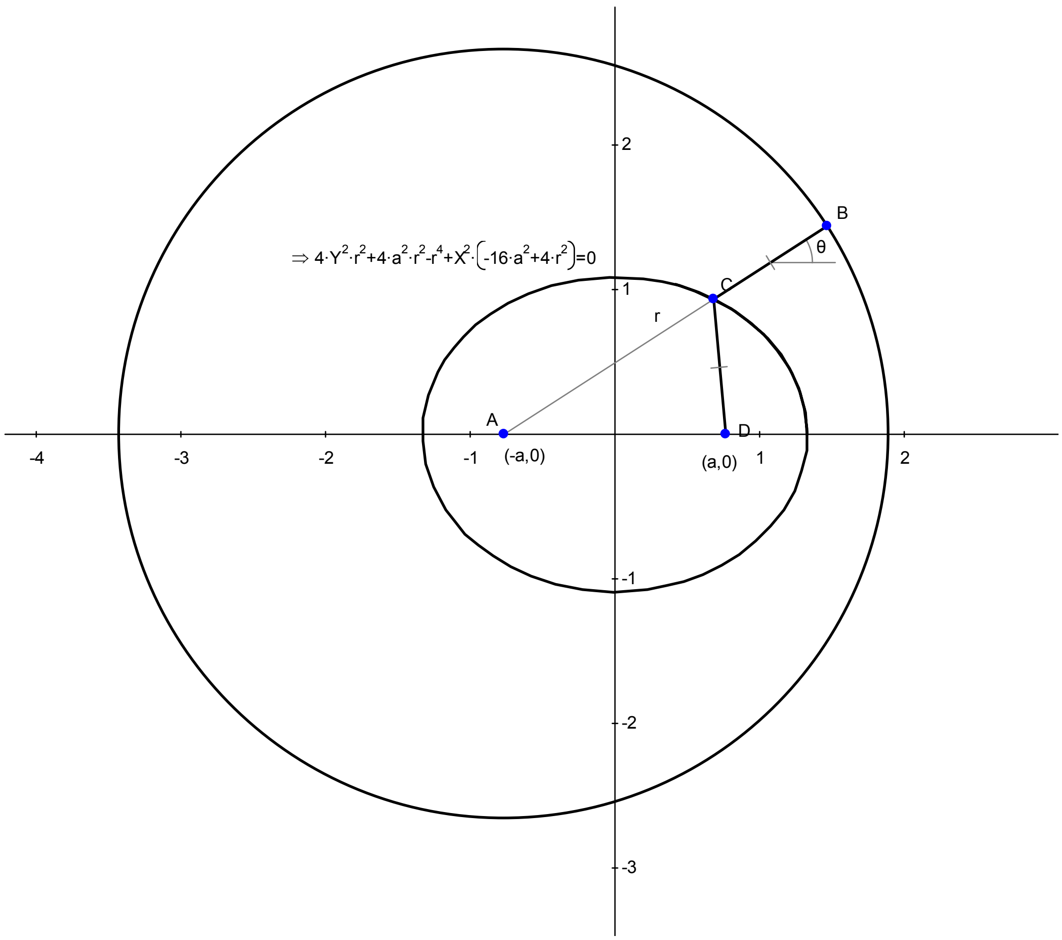
Example 8: An Alternative Ellipse Construction

Here is a construction (ascribed to Newton) which builds the ellipse from concentric circles radius equivalent to the semi major axes



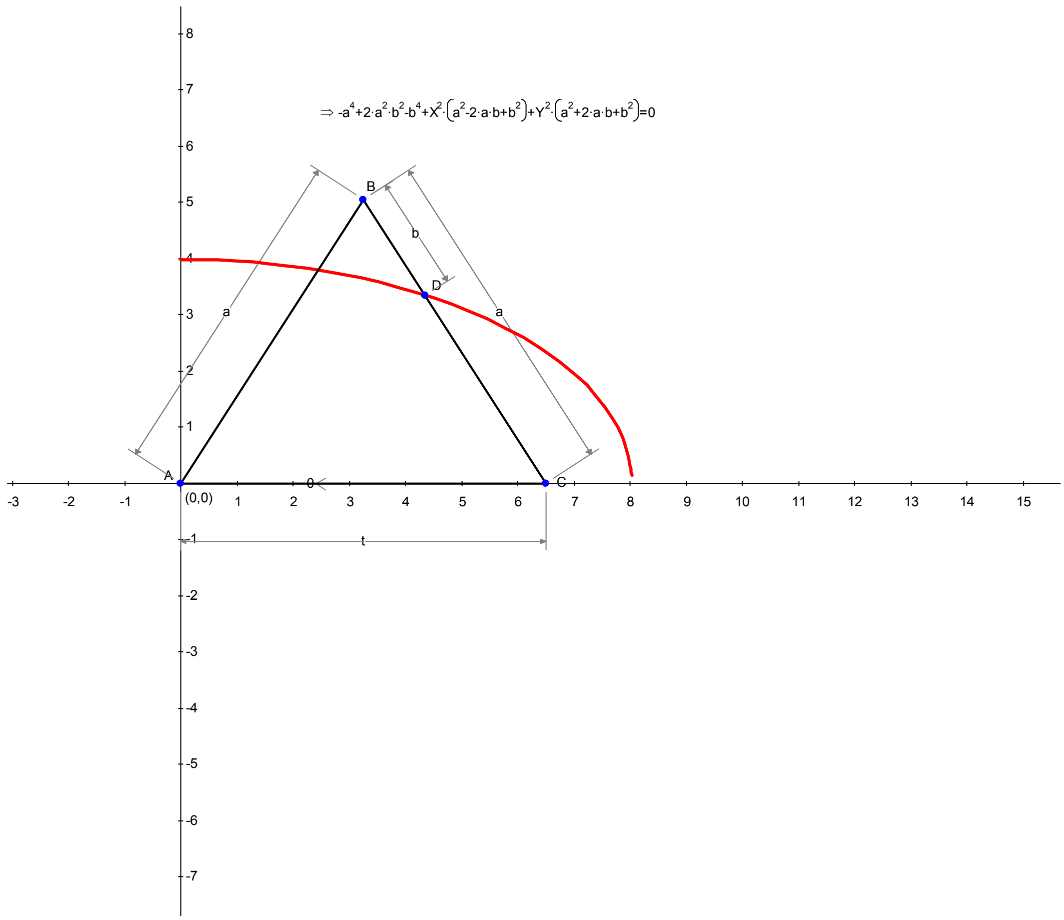
Example 9: Another ellipse

This time take a circle and a point, and the location of all points equidistant from the circle and the point:



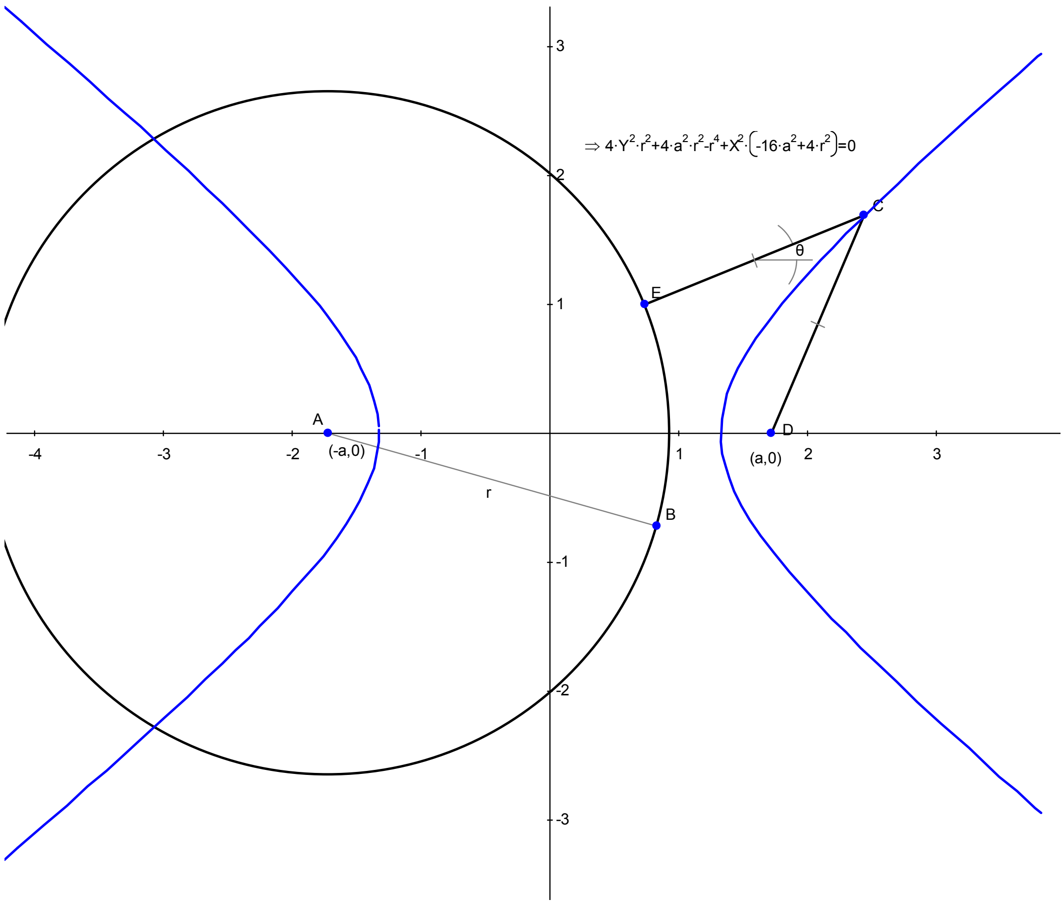
Example 10: "Bent Straw" Ellipse Construction

Here is another ellipse construction. Geometrically observe that the semi major axes are x-a and x+a. Can you verify this from the algebraic expression?



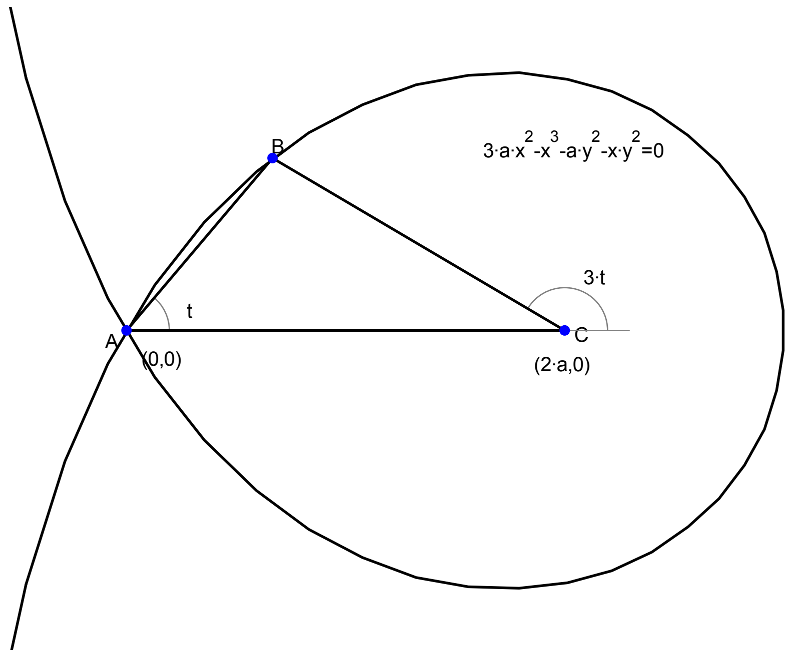
Example 11: Similar construction for a Hyperbola

If we do a similar construction, with the generating point outside the circle, we get a hyperbola:

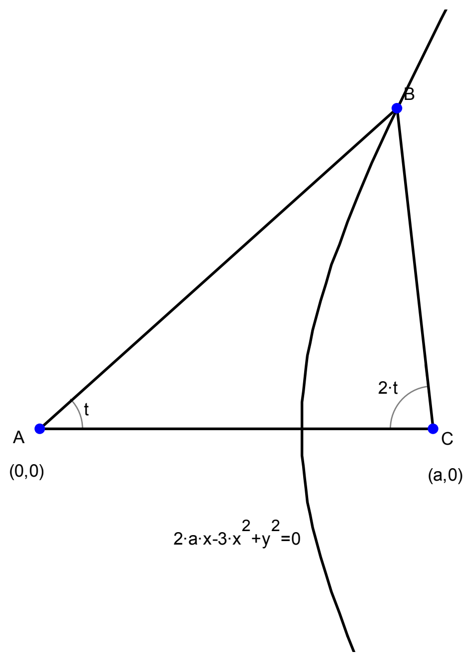


Example 12: Hyperbola Using MacLaurin's Trisectrix like construction

A cubic derived from the intersection of two lines rotating at different speeds

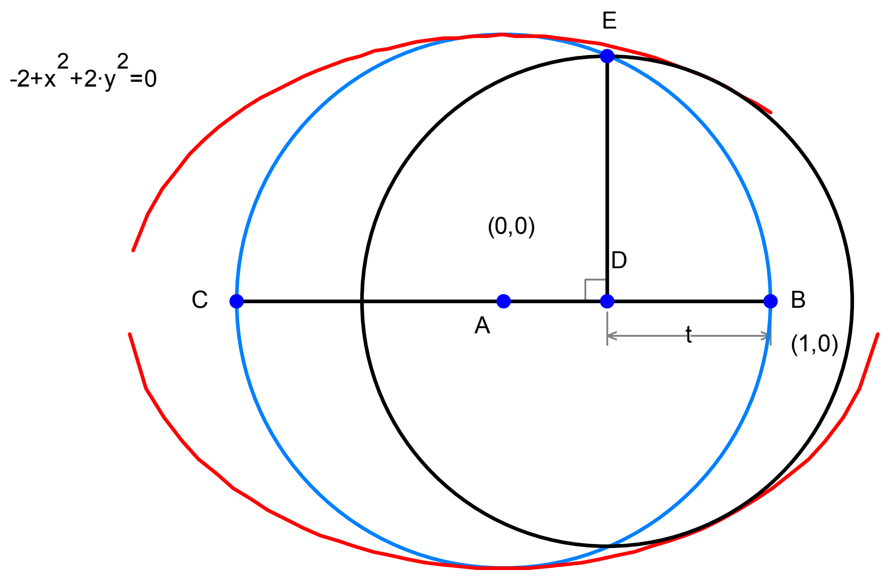


A similar construction can give a range of other curves. For example, a hyperbola:



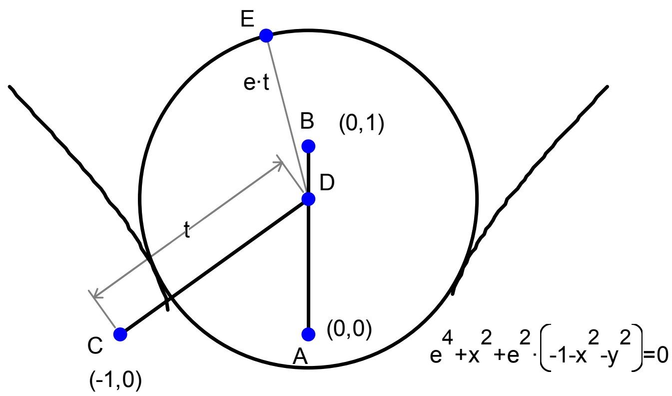
Example 13: Ellipse as Envelope of Circles

Take the envelope of the circles whose centers lie on the x-axis and which have extrema which lie on the unit circle. We find it is an ellipse:



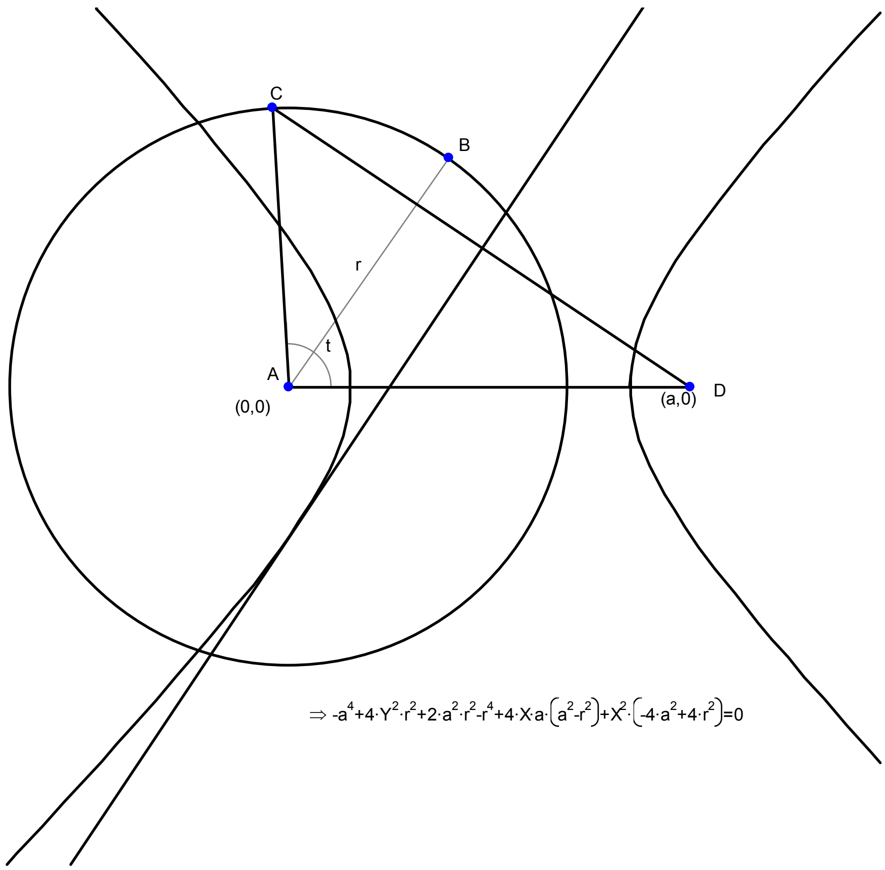
Example 14: Hyperbola as an envelope of circles

Take the envelope of a family of circles centered on a line and whose radius is an eccentricity times the distance from a focus.



Example 15: Hyperbola as an Envelope of Lines

We take the envelope of the perpendicular bisectors of the line CD as C traverses the circle AB.



The result is a hyperbola with foci A and B.

What happens if D lies inside the circle?